Goal / Outline

1. Poincaré Inequality
2. Difference Quotients

Notation

$(u)_\Omega = ⨍_\Omega u\,dy$ = average of u over Ω

$(u)_{x,r} = ⨍_{B(x,r)} u\,dy$ = average of u over the ball $B(x,r)$

Previous Theorems which will be used

Rellich-Kondrachov Compactness Theorem (Evans, §5.7, Theorem 1)

Assume U is a bdd open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1 \le p < n$. Then

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \le q < p^*$, with $p^* = \frac{pn}{n-p}$.

Remark (after proof of RKC Theorem)

$$W^{1,p}(V) \subset\subset L^p(V) \qquad \forall\ 1 \le p < \infty.$$

Theorem (Global approximations by smooth functions) (Evans, §5.2, Theorem 2)

Assume U is , and suppose as well that $u \in W^{k,p}(U)$ for some $1 \le p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_m \longrightarrow u \quad \text{in} \quad W^{k,p}(U).$$

Theorem (Weak Compactness) (Evans D.4)

Let X be a reflexive Banach space and suppose the seq. $\{u_k\}_{k=1}^\infty \subset X$ is bdd. Then there exists a subseq. $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$ and $u \in X$ s.t.

$$u_{k_j} \longrightarrow u.$$

Theorem 1 (Poincaré's inequality)

Let Ω be a bdd, connected, open subset of $\mathbb{R}^n$, w/ a C^1 boundary $\partial\Omega$. Assume $1 \le p \le \infty$. Then there exists a constant C, depending only on n, p, and Ω, such that

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le C \|Du\|_{L^p(\Omega)}$$

for each function $u \in W^{1,p}(\Omega)$.

This is nice b/c gradient of u only appears on the RHS. Similar to Gagliardo-Nirenberg-Sobolev, main differences are in GNS $p < \infty$ but in Poincaré we additionally require C^1 boundary and Ω to be connected. GNS is discussed in thms 1, 2, & 3 in §5.6.1.

Proof (By Contradiction)

Assume the statement is false, then for each $k = 1, \dots$ there exists a function $u_k \in W^{1,p}(\Omega)$ satisfying

$$\|u_k - (u)_\Omega\| > k \|Du_k\|_{L^p(\Omega)}. \quad (1)$$

Define $v_k \in L^p(\Omega)$ as

$$v_k := \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}$$

for $k = 1, \dots$

Then,

$$(v_k)_\Omega = \frac{1}{\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}} ⨍_\Omega v_k-(v_k)_\Omega \,dy$$
$$= \frac{1}{\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}} \left(⨍_\Omega v_k \,dy - (v_k)_\Omega \right)$$
$$= 0$$

and $\|v_k\|_{L^p(\Omega)} = 1$, therefore from 1

$$\|Dv_k\|_{L^p(\Omega)} = \frac{\|D(u_k-(u_k)_\Omega)\|_{L^p(\Omega)}}{\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}} = \frac{\|Du_k\|_{L^p(\Omega)}}{\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}} < \frac{\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}}{k\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}} = \frac{1}{k}.$$

In particular, $\|Dv_k\|_{L^p(\Omega)} < \frac{1}{k}$ 2 and $\{v_k\}_{k=1}^\infty$ are bdd in $W^{1,p}(\Omega)$.

Therefore from Rellich-Kondrachov Theorem (this is why we requrie C^1 boundary) since $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ (for all $1\leq p\leq\infty$) (Remark after theorem of R-K Thm) and $\{v_k\}$ bdd in $W^{1,p}(\Omega)$ there exists a subseq. $\{v_{k_j}\}_{j=1}^\infty \subset \{v_k\}_{k=1}^\infty$ and a fct $v \in L^p(\Omega)$ such that

$$v_{k_j} \longrightarrow v \quad \text{in} \quad L^p(U).$$

Therefore,

$$(v)_U = 0 \quad \text{and} \quad \|v\|_{L^p(\Omega)} = 1.$$

However,

$$\int_\Omega v\,\phi_{x_i}\,dx = \lim_{k_j\to\infty} \int_\Omega v_{k_j}\phi_{x_i}\,dx$$
$$= -\lim_{k_j\to\infty}\int_\Omega v_{k_j,x_i}\,\phi\,dx$$
$$= -\int_\Omega \underbrace{\lim_{k_j\to\infty} v_{k_j,x_i}}_{=0 \text{ by } 2}\,\phi\,dx = 0.$$

Therefore, $v \in W^{1,p}(\Omega)$ w/ $Dv = 0$ a.e. and since Ω is connected this implies v is constant. Recall $(v)_\Omega = 0$ which (along w/ v constant) implies $v \equiv 0$, which contradicts $\|v\|_{L^p(\Omega)} = 1$.

$\square$

Notation

$$(u)_{x,r} = ⨍_{B(x,r)} u \, dy$$

Theorem 2 (Poincaré's inequality for a ball)

Assume $1 \le p \le \infty$. Then there exists a constant C, depending only on n & p, such that

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \le Cr \|Du\|_{L^p(B(x,r))}$$

for each ball $B(x,r) \subset \mathbb{R}^n$ and each fct $u \in W^{1,p}(B^\circ(x,r))$.

Proof

The case $\Omega = B^\circ(0,1)$ follows from thm 1.

In general, if $u \in W^{1,p}(B^\circ(x,r))$, define

$$v(y) := u(x+ry)$$

so that $y \in B(0,1)$. Then $v \in W^{1,p}(B^\circ(0,1))$ and from thm 1 we have

$$\|v - (v)_{0,1}\|_{L^p(B(0,1))} \le C \|Dv\|_{L^p(B(0,1))}.$$

Then changing variables back to $B(x,r)$ and noting $Dv(y) = r\,Du(x+ry)$ we get

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \le Cr \|Du\|_{L^p(B(x,r))}.$$

$\square$

Poincaré Application Example

Consider the elliptic PDE & bdd open $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$, the variational formulation of this problem is to find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) = L(v)$$

for all $v \in H_0^1(\Omega)$, where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \& \quad L(v) = \int_\Omega f v \, dx.$$

Lax-Milgram

Let H be a Hilbert space and $a: H \times H \to \mathbb{R}$ a bilinear form, if a is bdd and coercive then there exists a unique sol'n $u \in H$ to the variational formulation

$$a(u,v) = L(v) \qquad \forall v \in H$$

where $L \in H'$, and $\|u\|_H \leq \frac{\|L\|_{H'}}{b}$.

$H_0^1(\Omega)$ is a Hilbert space &

(bounded) $|a(u,v)| = |\int \nabla u \cdot \nabla v \, dx| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$ ✓

(coercive) $\|v\|^2_{H_0^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$

$\leq C^2 \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$

Poincaré on $W_0^{1,p}$, $1 \leq p \leq \infty$, $\Omega \subset \mathbb{R}^n$ bdd & open
Evans, Thm 3, § 5.6.1.

$$\Rightarrow \quad |a(v,v)| = \|\nabla v\|^2_{L^2(\Omega)} \geq \frac{1}{C^2+1} \|v\|^2_{L^2(\Omega)} \quad ✓$$

Therefore, by Lax-Milgram $\exists!$ solution $u \in H_0^1$ to the variational formulation & furthermore

$$\|u\|_{H_0^1} \leq (C^2+1) \|L\|_{(H_0^1)'}$$

Difference Quotients

Will use difference quotient approximations when studying regularity of weak solutions to 2^{nd} order elliptic PDEs.

Assume $u: \Omega \to \mathbb{R}$ is a locally summable fct and $V \subset\subset \Omega$.

Definitions (Difference Quotients)

(i) The i^{th} **difference quotient** of size h is

$$D_i^h u(x) = \frac{u(x+he_i) - u(x)}{h} \qquad \text{for } i=1,\dots,n$$

for $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < \text{dist}(V, \partial\Omega)$

(ii) $D^h u := (D_1^h u, \dots, D_n^h u)$.

Theorem 3 (Difference quotients and weak derivatives)

(i) Suppose $1 \le p < \infty$ and $u \in W^{1,p}(\Omega)$. Then for each $V \subset\subset \Omega$

$$\|D^h u\|_{L^p(V)} \le C \|Du\|_{L^p(\Omega)}$$

for some constant C and all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial\Omega)$.

(ii) Assume $1 < p < \infty$, $u \in L^p(V)$, and there exists a constant C such that

$$\|D^h u\|_{L^p(V)} \le C$$

for all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial\Omega)$. Then

$$u \in W^{1,p}(V), \quad w/ \ \|Du\|_{L^p(V)} \le C$$

Proof

(i) Assume $1 \le p < \infty$, & assume u is smooth (this assumption will be relaxed later via global approximation by smooth functions). Then for each $x \in V$, $i=1,\dots,n$, and $0<|h|<\frac{1}{2}\operatorname{dist}(V,\partial\Omega)$, by fundamental theorem of calculus we have

$$u(x+he_i)-u(x) = h\int_0^1 u_{x_i}(x+the_i)\,dt$$

$$\Rightarrow |u(x+he_i)-u(x)| \le |h|\int_0^1 |u_{x_i}(x+the_i)|\,dt$$

$$\le |h|\int_0^1 |Du(x+the_i)|\,dt$$

Therefore, $|D^h u|^p \le C\sum_{i=1}^n |D_i^h u|^p$, by equivalence of norms

$$\int_V |D^h u|^p dx \le \int_V C\sum_{i=1}^n \left|\frac{u(x+he_i)-u(x)}{h}\right|^p dx$$

$$\le \sum_{i=1}^n \int_V \left(\int_0^1 |Du(x+the_i)|\right)^p dt\, dx$$

$$\le \sum_{i=1}^n \int_V\int_0^1 |Du(x+the_i)|^p\, dt\, dx$$

$$= \sum_{i=1}^n \int_0^1\int_V |Du(x+the_i)|^p dx\, dt$$

$$\le C\int_\Omega |Du|^p dx$$

Jensen's inequality which is valid since $(\cdot)^p$ is convex and Du is locally summable (b/c $u \in W^{1,p}(V)$ by assumption). (Evans, B.1, Theorem 2)

Therefore,

$$\int_V |D^h u|^p dx \le C\int_\Omega |Du|^p dx$$

if $u \in W^{1,p}(V)$ is smooth. Consider $\hat{u} \in W^{1,p}(\Omega)$ arbitrary, there exist fcts $u_m \in C^\infty(\Omega)\cap W^{1,p}(\Omega)$ such that

$$u_m \longrightarrow u \quad \text{in} \quad W^{1,p}(u).$$

Since u_m are smooth we have

$$\|D^h u_m\|_{L^p(V)} \le C\|Du_m\|_{L^p(\Omega)}$$

let $m \to \infty$ and we have

$$\|D^h u\|_{L^p(V)} \le C\|Du\|_{L^p(\Omega)} \qquad \square$$

(ii) Suppose

$$\|D^h u\|_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$ and some constant C. For $i = 1, \dots, n$, $\phi \in C_c^\infty(V)$, and small enough h (so $x + he_i \in V$ & $x - he_i \in V$) we have

$$\int_V u(x) \left[\frac{\phi(x+he_i) - \phi(x)}{h} \right] dx = \frac{1}{h} \left(\int_V u(x) \phi(x+he_i) dx - \int_V u(x) \phi(x) dx \right)$$

$$= \frac{1}{h} \left(\int_V u(x-he_i) \phi(x) dx - \int_V u(x) \phi(x) dx \right)$$

$$= -\int_V \left[\frac{u(x) - u(x-he_i)}{h} \right] \phi(x) \, dx.$$

that is

$$\int_V u (D_i^h \phi) \, dx = -\int_V (D_i^{-h} u) \phi \, dx$$

this is "integration-by-parts" for difference quotients.

Note that

$$\|D^h u\|_{L^p(V)} \leq C \text{ for all } 0 < |h| < \tfrac{1}{2} dist(V, \partial\Omega)$$

$$\Rightarrow \sup_h \|D_i^{-h} u\|_{L^p(V)} < \infty.$$

Therefore, from weak compactness (Theorem 3, D.4) and the fact that $L^p(V)$ is reflexive if $1 < p < \infty$ there exists a fct $v_i \in L^p(V)$ and a subseq $h_k \to 0$ such that

$$D_i^{-h_k} u \longrightarrow v_i \quad \text{weakly in } L^p(V).$$

Then

$$\int_V u \phi_{x_i} dx = \int_\Omega u \phi_{x_i} dx$$

$$= \lim_{h_k \to 0} \int_\Omega u D_i^{h_k} \phi \, dx$$

$$= -\lim_{h_k \to 0} \int_V D_i^{-h_k} u \, \phi \, dx$$

$$= -\int_V v_i \phi \, dx$$

$$= -\int_\Omega v_i \phi \, dx.$$

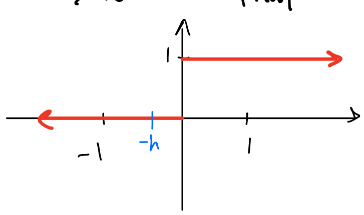
Therefore $v_i = u_{x_i}$ in the weak sense for $i=1,\dots,n$ and since $v_i \in L^p(V)$ this implies $Du \in L^p(V)$. Along w/ the assumption that $u \in L^p(V)$, we conclude $u \in W^{1,p}(V)$.

Then
$$\|Du\|_{L^p(V)} = \|v\|_{L^p(V)}$$
where $D^{-h_k}u = v$, also
$$\|D^{-h_k}u\|_{L^p(V)} \le C$$
by assumption. Let $h_k \to 0$ to get
$$\|Du\|_{L^p(V)} = \|v\|_{L^p(V)} \le C$$
□

Example (for assertion (ii) of Theorem 3 being false if $p=1$)

Let $u = \begin{cases} 1 & \text{if } x \in (0,-1) \\ 0 & \text{if } x \in (-1,0) \end{cases}$

then consider fixed but arbitrary h such that $0 < h < \frac{1}{2}\operatorname{dist}(V, \partial\Omega)$ then



$$D^h u = \frac{u(x+h) - u(x)}{h} = \begin{cases} \frac{1}{h} & \text{if } -h < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

Then
$$\|D^h u\|_{L^1((-1,1))} = \int_{-1}^{1} |D^h u|\, dx = \int_{-h}^{0} \frac{1}{h}\, dx = 1$$
therefore satisfies $\|D^h u\|_{L^1(V)} \le 1$ but has no weak derivative, therefore $u \notin W^{1,1}(V)$.

Remark

Variants of Thm 3 can hold true w/out $V \subset\subset U$.

For example $\Omega = B^0(0,1) \cap \{x_n > 0\}$, $V = B^0(0,\frac{1}{2}) \cap \{x_n > 0\}$,

we have the bound $\int_V |D_i^h u|^p dx \leq \int_\Omega |u_{x_i}|^p dx$ for $i = 1, \dots, n-1$.